Worcester County Mathematics League

Varsity Meet 3 - January 24, 2024

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 3 - January 24, 2024
Answer Key

Round 1 Similarity and Pythagorean Theorem

1. 14
2. $\frac{5\sqrt{43}}{2}$
3. $\frac{24}{35}$

Round 2 - Algebra I

1. 10
2. 6
3. -4

Round 3 - Functions

1. -3
2. $\{5 \leq x \leq 7\}$ or
   $\{x \geq 5 \text{ AND } x \leq 7\}$ or
   $\{x \geq 5\} \bigcap \{x \leq 7\}$ or
   $[5, 7]$ (interval notation was accepted at the in person meet; also note that curly brackets are provided in answer space for this question)
3. 8

Round 4 - Combinatorics

1. 120
2. 25350
3. 44

Round 5 - Analytic Geometry

1. -3
2. 33
3. $\left(\frac{5}{2}, \frac{1}{2}\right)$ or $(2.5, 0.5)$ or $\left(2\frac{1}{2}, \frac{1}{2}\right)$ (exact order)

Team Round

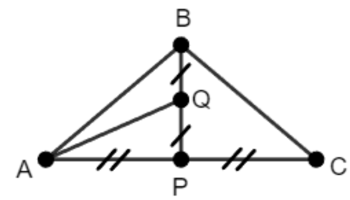
1. $\frac{20}{3}$
2. 245
3. $\frac{3+\sqrt{5}}{2}$
4. 1260
5. $(4, -2)$ (exact order)
6. $\frac{20}{121}$
7. 215
8. $\frac{9}{11}$
9. $-1, 9$ (need both, either order)

Worcester County Mathematics League
Varsity Meet 3 - January 24, 2024
Round 1 - Similarity and Pythagorean Theorem

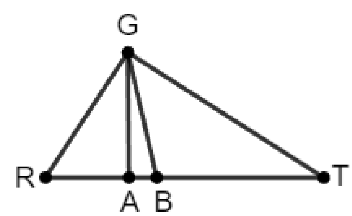
All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. The sum of the areas of two similar pentagons is 700. If the ratio of the lengths of their corresponding sides is 1:7, find the area of the smaller pentagon.

2. Triangle ABC, shown at right, has sides of lengths $AC = 30$ and $AB = BC = 20$. If P is on $\overline{AC}$ such that $\overline{BP}$ is a median of $\triangle ABC$ and Q is the midpoint of $\overline{BP}$, find AQ.



3. Given A and B on $\overline{RT}$ of $\triangle RGT$ as shown at right, where $m\angle RGT = 90°$, $\overline{GA} \perp \overline{RT}$, $GA = 4\frac{4}{5}$, $RG = 6$, and $\overline{GB}$ bisects $\angle RGT$; find AB.



ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. $AQ =$ ______________________

(3 pts) 3. $AB =$ ______________________

Groton-Dunstable, St. John's, Tahanto

Worcester County Mathematics League
Varsity Meet 3 - January 24, 2024
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Aldore is 8 years older than his sister Cherchy. In 6 years Aldore will be twice the age of Cherchy. How old is Aldore now?

2. A bronze shield with a ratio of tin to copper equal to 3 : 5 is melted with a bronze sword with a tin to copper ratio of 1 : 2. The resulting metal contains 5 pounds (lbs) of tin and 9 pounds (lbs) of copper. How much did the sword weigh, in lbs?

3. Solve the following equation for x:

$$\frac{2x+20}{x+8} = \frac{x+10}{x+2} - \frac{48}{x^2+10x+16}$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________ lbs

(3 pts) 3. $x =$ ____________________

Auburn, Shepherd Hill, Clinton/QSC

Worcester County Mathematics League
Varsity Meet 3 - January 24, 2024
Round 3 - Functions

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $f(x) = 2x + 1$ and $g(x) = 3x - 2(2x + 4)$, for what value of x does $f(x) = g(x)$?

2. Let S be the subset of Real numbers that are excluded from the domain of $f(x) = \frac{1}{\sqrt{x^2-12x+35}}$. Express S using set notation.

3. Given $f^{-1}(x) = h(x)$, $g^{-1}(x) = \frac{16}{h(x)}$, and $f(x) = \sqrt[3]{x-3}$, find $h(g(2))$.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. $S = \{$________________$\}$

(3 pts) 3. ________________

Shepherd Hill, Quaboag, Shrewsbury

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. A keypad has the digits 0-5 on it. How many 3 digit sequences are possible if you can only touch each button once?

2. From a standard 52 card deck, with 13 spades, 13 hearts, 13 diamonds, and 13 clubs, how many four card combinations can be formed of two black cards and two hearts? Note that clubs and spades are black, hearts and diamonds are red, and all cards are distinct.

3. Five friends, Amy, Bill, Clark, Deborah, and Edna, each bring a different hat to a party. When they leave, they each take home someone else's hat! How many different ways could that happen (the five friends each take someone else's hat)?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

St. John's, St. Peter-Marian, AMSA/QSC

Worcester County Mathematics League

Varsity Meet 3 - January 24, 2024

Round 5 - Analytic Geometry

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Find the value for k for which the line $4x + ky = 5$ passes through the point $(x, y) = (2, 1)$.

2. What is the greatest distance between a point on the circle with equation $x^2 + y^2 + 2x - 10y - 23 = 0$ and a point on the circle with equation $x^2 + y^2 - 16x + 14y - 8 = 0$?

3. Find the point (x, y) that is equidistant from the three points $(0, 0)$, $(2, 3)$, and $(3, -2)$.

ANSWERS

(1 pt) 1. $k =$ ________________

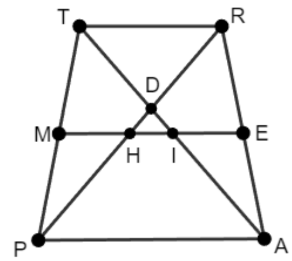
(2 pts) 2. ________________

(3 pts) 3. $(x, y) =$ (________________)

Shepherd Hill, St. John's, Shepherd Hill

Worcester County Mathematics League
Varsity Meet 3 - January 24, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED



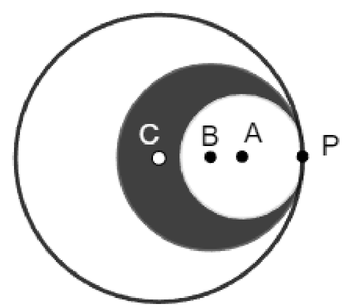
1. $\overline{ME}$ is the median of trapezoid $TRAP$, as shown at right. Diagonals $\overline{TA}$ and $\overline{RP}$ intersect at D within quadrilateral $TREM$. $\overline{RP}$ intersects $\overline{ME}$ at H and $\overline{TA}$ intersects $\overline{ME}$ at I. If $TR = 4$, $TD = 3$, and $DI = 1$, find PA. Express your answer as a simplified improper fraction in the form $\frac{m}{n}$.

2. Mr. Smith can grade a batch of quizzes in three hours. Mr. Jones can grade a batch of quizzes in two and a half hours. If they work together at the above rates, how long will it take them to grade 3 batches of quizzes? Express your answer rounded to the nearest number of minutes.

3. Given the function $f(x) = \frac{1}{x}$, find the largest value of d that satisfies the equation $f(d) + f\left(\frac{1}{d}\right) = 3$. Express your answer in the form $\frac{a+\sqrt{b}}{c}$.

4. The Groton-Dunstable A Capella club has 3 sophomores, 6 juniors, and 8 seniors. How many different five person groups can be formed from the members of the club if one sophomore, two juniors, and two seniors must be in a group?

5. The three vertices of $\triangle ABC$ have coordinates $A(3, 8)$, $B(10, -6)$, and $C(-1, -8)$. Find (x, y), the coordinates of the point of intersection of the three medians of $\triangle ABC$.



6. Circles A, B, and C are internally tangent at point P, as shown at right. If $r_A = \frac{2}{3}r_B$ and $r_B = \frac{6}{11}r_C$, where r_A, r_B, and r_C are the radii of circles A, B, and C. Find $\frac{A_S}{A_C}$, the ratio of the shaded region (A_S) to the area of circle C (A_C), expressed as a fraction.

7. Let $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^2 + 2$. Find the value of $f^{-1}\left(g\left(f(-9)\right)\right)$.

8. There are 5 green marbles and 6 red marbles in a bag. Three marbles randomly chosen from the bag. What is the probability that one marble is a different color from the other two? Write your answer as a fraction $\frac{m}{n}$.

9. Circle A has a diameter with endpoints $(2, -7)$ and $(6, 3)$. The point $(x, -4)$ is on the circle. Find all possible values of x.

AMSA,Algonquin,Leicester,Groton-Dunstable,Assabet, Quaboag, Shrewsbury, Tantasqua/QSC, Quaboag

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. $PA =$ ____________________

2. ____________________ minutes

3. $d =$ ____________________

4. ____________________ groups

5. $(x, y) = ($____________________$)$

6. ____________________

7. ____________________

8. ____________________

9. $x =$ ____________________

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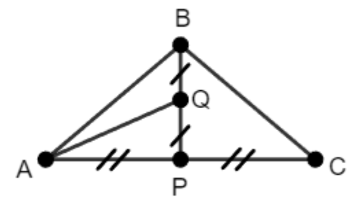
Round 1 - Similarity and Pythagorean Theorem

1. The sum of the areas of two similar pentagons is 700. If the ratio of the lengths of their corresponding sides is 1:7, find the area of the smaller pentagon.

Solution: Recall that if corresponding lengths of two similar polygons are in ratio $k : l$, then the areas of the two polygons are in ratio $k^2 : l^2$. Therefore the areas of the two pentagons are in ratio $(1^2 : 7^2) = (1 : 49)$. Let the area of the smaller pentagon be x. Then the area of the larger pentagon is $49x$. The sum of the two areas of the pentagons is $x + 49x = 50x = 700$. Solve for the area of the smaller pentagon: $x = \frac{700}{50} = \frac{70}{5} = \boxed{14}$.

2. Triangle ABC, shown at right, has sides of lengths $AC = 30$ and $AB = BC = 20$. If P is on $\overline{AC}$ such that $\overline{BP}$ is a median of $\triangle ABC$ and Q is the midpoint of $\overline{BP}$, find AQ.



Solution: Note that $\overline{AB} \cong \overline{BC}$ and therefore $\triangle APB \cong \triangle CPB$ by SSS. Then corrresponding angles $\angle APB \cong \angle CPB$ and $m\angle APB = m\angle CPB = 90°$ because these angles are a linear pair with equal measures.

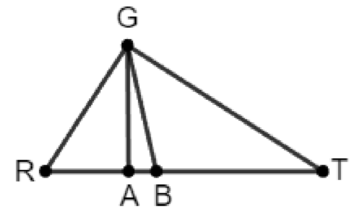
Note that $AP = CP = \frac{AC}{2} = 15$ and that $PQ = BQ = \frac{BP}{2}$. Apply the Pythagorean Theorem to right $\triangle APB$ to find BP:

$$\begin{aligned} AP^2 + BP^2 &= AB^2 \\ (15)^2 + BP^2 &= (20)^2 \\ BP^2 &= (20)^2 - (15)^2 = 400 - 225 = 175 \\ BP &= \sqrt{175} = \sqrt{25 \cdot 7} = \sqrt{25}\sqrt{7} = 5\sqrt{7} \end{aligned}$$

Now $PQ = \frac{BP}{2} = \frac{5}{2}\sqrt{7}$ and $\triangle APQ$ is also a right triangle because of right angle $\angle APQ$. Apply the Pythagorean Theorem once more:

$$\begin{aligned} AQ^2 &= AP^2 + PQ^2 \\ &= 15^2 + (\frac{5}{2}\sqrt{7})^2 \\ &= 225 + \frac{25 \cdot 7}{4} = \frac{225 \cdot 4 + 175}{4} \\ &= \frac{900 + 175}{4} = \frac{1075}{4}. \end{aligned}$$

Then $AQ = \sqrt{\frac{1075}{4}} = \frac{\sqrt{25 \cdot 43}}{2} = \boxed{\frac{5\sqrt{43}}{2}}$.



3. Given A and B on $\overline{RT}$ of $\triangle RGT$ as shown at right, where $m\angle RGT = 90°$, $\overline{GA} \perp \overline{RT}$, $GA = 4\frac{4}{5}$, $RG = 6$, and $\overline{GB}$ bisects $\angle RGT$; find AB.

Solution: First convert GA to an improper fraction, $GA = \frac{24}{5}$. Note that $\triangle RAG$ is a 3-4-5 triangle since $RG = 6 = \frac{30}{5}$, so that $RA = \frac{3}{5} \cdot 6 = \frac{18}{5}$. Next, recall that the altitude of a right triangle divides the triangle into two similar triangles that are also similar to the original right triangle. Therefore $\triangle RGT \sim \triangle RAG$ and $\triangle RGT$ is also a 3-4-5 triangle. Leg $\overline{RG}$ of $\triangle RGT$ corresponds to leg $\overline{RA}$ of $\triangle RAG$, which is the smaller of the two legs. Because $RG = 6$, $\triangle RGT$ is a 3-4-5 triangle with a scale factor of 2, so that $GT = 8$ and $TR = 10$.

Apply the angle bisector theorem, which states that an segment that bisects the angle of a triangle divides the opposite sides in the same ratio as the ratio of the other two sides. Thus, $RB : BT = RG : GT = 6 : 8 = 3 : 4$. Then $RB = \frac{3}{7}RT$ and $BT = \frac{4}{7}RT$. Since $RT = 10$, $RB = \frac{30}{7}$ and $AB = RB - RA = \dfrac{30}{7} - \dfrac{18}{5} = \dfrac{5 \cdot 30 - 7 \cdot 18}{35} = \dfrac{150 - 126}{35} = \boxed{\dfrac{24}{35}}$

Round 2 - Algebra I

1. Aldore is 8 years older than his sister Cherchy. In 6 years Aldore will be twice the age of Cherchy. How old is Aldore now?

Solution: First assign variables. Let a be Aldore's current age and c be Cherchy's current age. Note that in 6 years Aldore's age will be $a+6$ and Cherchy's age will be $c+6$. Next, translate the given information into two equations:

$$a = c + 8$$
$$a + 6 = 2(c + 6).$$

Now there are two equations in two unknowns, where the first equation is equivalent to $c = a - 8$. Substitute $a - 8$ for c in the second equation and solve for a:

$$a + 6 = 2(a - 8 + 6) = 2(a - 2) = 2a - 4$$
$$6 + 4 = 2a - a = a$$

so $a = \boxed{10}$. Check the answer by noting that Aldore's and Cherchy's current ages are 10 and 2, and their ages in 6 years are 16 and 8, where $16 = 2 \cdot 8$.

2. A bronze shield with a ratio of tin to copper equal to 3 : 5 is melted with a bronze sword with a tin to copper ratio of 1 : 2. The resulting metal contains 5 pounds (lbs) of tin and 9 pounds (lbs) of copper. How much did the sword weigh, in lbs?

Solution: Let x be the weight of the sword and y be the weight of the shield, both in lbs. Then the shield contributes $\frac{3}{8}y$ lbs of tin and $\frac{5}{8}y$ lbs of copper to the metal. Likewise, the sword contributes $\frac{1}{3}x$ lbs of tin and $\frac{2}{3}x$ lbs of copper to metal. Set up a system of two equations in x and y by adding the contributions of the sword and the shield:

$$\frac{1}{3}x + \frac{3}{8}y = 5$$
$$\frac{2}{3}x + \frac{5}{8}y = 9$$

Multiply both sides of both equations by 24:

$$8x + 9y = 120$$
$$16x + 15y = 216$$

At this point it is perhaps easiest to eliminate x to solve for y, and then solve for x using the fact that $x + y = 5 + 9$ (the total weight of metal). Multiple the first equation by 2 ($16x + 18y = 240$) and subtract the second equation, so that $(18 - 15)y = 3y = 240 - 216 = 24$, $3y = 24$, and $y = 8$. Then since $x + y = 5 + 9 = 14$, $x = 14 - y = 14 - 8 = \boxed{6}$ lbs.

3. Solve the following equation for x:

$$\frac{2x+20}{x+8} = \frac{x+10}{x+2} - \frac{48}{x^2+10x+16}$$

Solution: First note that $(x+8)(x+2) = x^2+10x+16$, that is, the denominator of the third rational expression is equal to the product of the denominators of the other two rational expressions. Use this fact to express each of the rational expressions with a common denominator:

$$\frac{(2x+20)(x+2)}{(x+8)(x+2)} = \frac{(x+10)(x+8)}{(x+2)(x+8)} - \frac{48}{x^2+10x+16}$$
$$\frac{(2x+20)(x+2)}{x^2+10x+16} = \frac{(x+10)(x+8)}{x^2+10x+16} - \frac{48}{x^2+10x+16}$$

Next, collect the three expressions on the LHS (left hand side) of the equation and add them:

$$\frac{(2x+20)(x+2)}{x^2+10x+16} - \frac{(x+10)(x+8)}{x^2+10x+16} + \frac{48}{x^2+10x+16} = 0$$
$$\frac{(2x+20)(x+2)-(x+10)(x+8)+48)}{x^2+10x+16} = 0$$

Note that x solves the equation when the numerator is equal to zero and the denominator is not zero. Because the denominator factors into $x^2+10x+16 = (x+8)(x+2)$, x cannot equal -8 or -2. Next, set the numerator equal to zero and solve, noting that $2x+20 = 2(x+10)$ and factoring $(x+10)$ out of two polynomials:

$$\begin{aligned}
(2x+20)(x+2)-(x+10)(x+8)+48 &= 0\\
2(x+10)(x+2)-(x+10)(x+8)+48 &= 0\\
(x+10)(2(x+2)-(x+8))+48 &= 0\\
(x+10)(2x+4-x-8))+48 &= 0\\
(x+10)(x-4)+48 &= 0\\
x^2+6x-40+48 &= 0\\
x^2+6x+8 &= 0
\end{aligned}$$

Now factor the quadratic: $(x+4)(x+2) = 0$. Both $x=-4$ and $x=-2$ solve this equation. However, $x=-2$ is an extraneous solution because it causes two denominators to equal zero in the original equation. Thus, $x = \boxed{-4}$.

Round 3 - Functions

1. If $f(x) = 2x + 1$ and $g(x) = 3x - 2(2x + 4)$, for what value of x does $f(x) = g(x)$?

Solution: Equate the two functions: $f(x) = 2x+1 = 3x-2(2x+4) = g(x)$. Expand and simplify the expression for $g(x)$: $3x - 2(2x + 4) = 3x - 4x - 8 = -x - 8$. Thus $2x + 1 = -x - 8$. Add x and subtract 1 from both sides and $3x = -9$. Divide by 3 and $\boxed{x = -3}$

2. Let S be the subset of Real numbers that are excluded from the domain of $f(x) = \frac{1}{\sqrt{x^2-12x+35}}$. Express S using set notation.

Solution: Recall that a number is excluded from the domain of a function if, when the function is evaluated for that number either:

- the denominator of a rational expression has the value 0 (zero) OR
- the expression under a square root (in general, any even root) has a negative value

In this case, the values for x that are excluded from the domain of $f(x)$ are the values for x for which $x^2 - 12x + 35 \leq 0$, since this polynomial is both the denominator of a rational expression and under a square root. Factor this polynomial: $p(x) = x^2 - 12x + 35 = (x - 5)(x - 7)$. Thus $p(x) = 0$ when $x = 5$ or $x = 7$.

Next, note that $p(x) = (x-5)(x-7)$ is negative when one term is positive and the other is negative. When x is between 5 and 7: $x - 5$ is positive and $x - 7$ is negative. Note that if x is less than 5, both terms are negative and if x is greater than 7, both terms are positive. Thus the answer is $\boxed{\{5 \leq x \leq 7\}}$, or more precisely $\{x : 5 \leq x \leq 7\}$.

3. Given $f^{-1}(x) = h(x)$, $g^{-1}(x) = \frac{16}{h(x)}$, and $f(x) = \sqrt[3]{x-3}$, find $h(g(2))$.

Solution: Apply $f(x)$ to both sides of $f^{-1}(x) = h(x)$: $f(f^{-1}(x)) = x = f(h(x))$, or $f(h(x)) = x$. Then apply the given definition of $f(x)$ so that $f(h(x)) = \sqrt[3]{h(x)-3} = x$. Cube both sides of the last equation: $\left(\sqrt[3]{h(x)-3}\right)^3 = h(x) - 3 = x^3$. Then $h(x) = x^3 + 3$.

Now $g^{-1}(x) = \frac{16}{h(x)} = \frac{16}{x^3+3}$. Recall that $g^{-1}(g(x)) = x$ so $\frac{16}{(g(x))^3+3} = x$. Multiply both sides by $\frac{(g(x))^3+3)}{x}$: $\frac{16}{x} = (g(x))^3 + 3$. Subtract 3 from both sides: $\frac{16}{x} - 3 = (g(x))^3$, or $(g(x))^3 = \frac{16}{x} - 3$. Then take the cube root: $g(x) = \sqrt[3]{\frac{16}{x} - 3}$.

Finally, evaluate $h(g(2))$:

$$g(2) = \sqrt[3]{\frac{16}{2} - 3} = \sqrt[3]{8-3} = \sqrt[3]{5}$$
$$h(g(2)) = h\left(\sqrt[3]{5}\right) = \left(\sqrt[3]{5}\right)^3 + 3 = 5 + 3 = \boxed{8}$$

Round 4 - Combinatorics

1. A keypad has the digits 0-5 on it. How many 3 digit ordered sequences are possible if you can only touch each button once?

Solution: The number of possible orderings of three digits chosen from six (0,1,2,3,4,5) digits can be calculated by considering the number of choices for the first, then the second, then the third digit. There are six choices for the first digit. Regardless of the choice for the first digit, there five choices for the second digit because one digit has already been chosen. Likewise, there are four choices for the third digit once the first two digits are chosen. In all there are $6 \cdot 5 \cdot 4 = 6 \cdot 20 = \boxed{120}$ possible three digit sequences.

Note that the answer to this problem is the number of permutations of 3 distinct elements chosen from a set of 6, or ${}_6P_3 = \frac{6!}{(6-3)!}$.

2. From a standard 52 card deck, with 13 spades, 13 hearts, 13 diamonds, and 13 clubs, how many four card combinations can be formed of two black cards and two hearts? Note that clubs and spades are black, hearts and diamonds are red, and all cards are distinct.

Solution: Let N_b be the number of ways to choose two black cards and N_h be the number of ways to choose two hearts. These two choices are independent and determine the four card hand, so the total number of hands, n, is the product $n = N_b \cdot N_h$.

There are 26 black cards and 13 hearts in the 52 card deck. Then $N_b = \binom{26}{2} = 26 \cdot 25/2$ (two black cards chosen from 26) and $N_h = \binom{13}{2} = 13 \cdot 12/2$ (two hearts chosen from 13). Finally, $n = N_b \cdot N_h = \frac{26 \cdot 25}{2} \cdot \frac{13 \cdot 12}{2} = 13 \cdot 25 \cdot 13 \cdot 6 = 13^2 \cdot 25 \cdot 6 = 169 \cdot 150 = \boxed{25350}$

3. Five friends, Amy, Bill, Clark, Deborah, and Edna, each bring a different hat to a party. When they leave, they each take someone else's hat! How many ways could the five friends each take home someone else's hat?

Solution: First consider Amy. She didn't take home her own hat, so there are 4 possible hats she could have taken home. Say that Amy took Bill's hat. Then either Bill either took Amy's hat or someone else's. If Bill took Amy's hat, then there are two ways to complete the choice of hats: Clark could have taken either Deborah or Edna's hat. In either case, Clark, Deborah, and Edna form a cycle (a graph theory concept). For instance, if Clark took Deborah's hat:

- Clark took Deborah's hat (out of two possible given that Amy's and Bill's were taken)
- Deborah took Edna's hat (not Clark's hat, or Edna would have taken her own hat)
- Edna took Clark's hat

To summarize, there are four hats that Amy might take home. If the person whose hat Amy took also took Amy's hat, then there are 2 possibilities, or $4 \cdot 2 = 8$ in total.

Now consider the case where the person whose hat Amy took, say Bill, took home one of the other three friend's hats, Clark's, Deborah's or Edna's. Say Bill took Clark's hat. If Clark then took Amy's hat, forming a cycle of three like above, then there is only one possibility: Deborah and Edna must have taken each other's hats. The total number of possible hat assignments for that case is $4 \cdot 3 = 12$, four choices for Amy and three choices of hats for the second person.

If instead Clark took Deborah's hat, then Deborah must have taken Edna's hat and Edna taken Amy's hat, forming a cycle of 5:

- Amy took Bill's hat (out of four possible)
- Bill took Clark's hat (out of three possible, if not Amy's)
- Clark took Deborah's hat (out of two possible, if not Amy's)
- Deborah took Edna's hat (not Amy's hat, or Edna would have taken her own hat)
- Edna took Amy's hat

In this case, there are $4 \cdot 3 \cdot 2 = 24$ possible hat assignments: four choices for Amy, three choices for the second person, and two choices for the third person. Summing the three cases, there are $8 + 12 + 24 = \boxed{44}$ ways the friends could have each taken the wrong hat home.

Alternative solution: Consider the number of ways of breaking up the group of 5 friends into cycles of length at least two. Either there will be be a pair and a trio, or there will be a cycle of length 5. If there is a pair and a trio, there are $\binom{5}{2}$ ways to choose the pair, and 2 ways to order the trio, so $\binom{5}{2}2 = 10 \cdot 2 = 20$ ways. The number of 5 person cycles equals the number of ways to order the 5 people divided by the number of ways to choose a first person of the cycle, or $5!/5 = 4! = 24$. Thus there are $20 + 24 = 44$ possibilities in total, as in the first solution.

Round 5 - Analytic Geometry

1. Find the value for k for which the line $4x + ky = 5$ passes through the point $(x, y) = (2, 1)$.

Solution: Since the point (2,1) is on the line, the values $x = 2$ and $y = 1$ will satisfy the equation of the line, $4x + ky = 5$. This means that $4 \cdot 2 + k \cdot 1 = 5$ and $8 + k = 5$ so $k = 5 - 8 = \boxed{-3}$

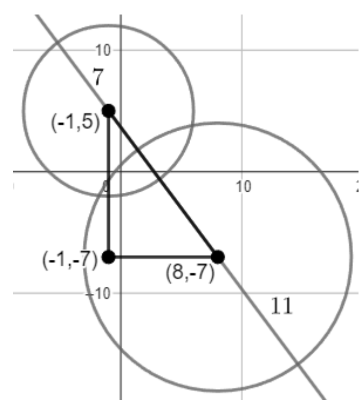
2. What is the greatest distance between a point on the circle with equation $x^2 + y^2 + 2x - 10y - 23 = 0$ and a point on the circle with equation $x^2 + y^2 - 16x + 14y - 8 = 0$?

Solution: Start by putting the two equations into standard form for a circle with center (x_0, y_0) and radius r $((x - x_0)^2 + (y - y_0)^2 = r^2)$ by completing the squares:

$$\begin{gathered} x^2 + 2x + y^2 - 10y - 23 = 0 \\ x^2 + 2x + 1 + y^2 - 10y + 25 = 1 + 25 + 23 = 49 \\ (x+1)^2 + (y-5)^2 = 7^2 \\ x^2 - 16x + y^2 + 14y - 8 = 0 \\ x^2 - 16x + 64 + y^2 + 14y + 49 = 8 + 64 + 49 = 121 \\ (x-8)^2 + (y+7)^2 = 11^2 \end{gathered}$$

Compare these equations to the standard form: the first circle has center $(-1, 5)$ and radius 7, and the second circle has center $(8, -7)$ and radius 11.

Sketch the two circles and a coordinate axis, as shown in the figure to the right. As can be seen, the greatest distance between points on the two circles is the length of a segment that lies on the line containing the two centers, specifically the segment whose endpoints are the furthermost intersection points of this line with the circles. This length is equal to the distance between the two centers plus the two radii.



The distance between the centers of the circles can be found by the distance formula:

$$\begin{aligned} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} &= \sqrt{(8 - (-1))^2 + (-7 - 5)^2} \\ &= \sqrt{9^2 + (-12)^2} \\ &= \sqrt{81 + 144} = \sqrt{225} = 15 \end{aligned}$$

Then the largest distance between points is equal to $15+7+11 = \boxed{33}$.

3. Find the point (x, y) that is equidistant from the three points $(0, 0)$, $(2, 3)$, and $(3, -2)$.

Solution: Name the points $A(0, 0)$, $B(2, 3)$, and $C(3, -2)$. Consider the three points as vertices of $\triangle ABC$. Note that any point on a perpendicular bisector of a line segment is equidistant from the endpoints of the segment. Thus, any point on the perpendicular bisector of $\overline{AB}$ is equidistant from A and B. Also, any point of the perpendicular bisector of $\overline{AC}$ is equidistant from A and C. Let $d(P, Q)$ be the distance between points P and Q. Then if P is on both of the two perpedicular bisectors $d(P, A) = d(P, B)$ and $d(P, A) = d(P, C)$, so $d(P, A) = d(P, B) = d(P, C)$, and P is equidistant from A, B, and C.

Find the point of intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ by first finding their equations. The perpendicular bisector of $\overline{AB}$ contains the midpoint of $\overline{AB}$ and has a slope that is the negative reciprocal of the slope of $\overline{AB}$. The midpoint of $\overline{AB}$ is $\left(\frac{0+2}{2}, \frac{0+3}{2}\right) = \left(1, \frac{3}{2}\right)$. The slope of $\overline{AB}$ is $\frac{3-0}{2-0} = \frac{3}{2}$. Therefore the slope of its perpendicular bisector is $-\frac{2}{3}$ (the negative reciprocal of $\frac{3}{2}$), and its equation is $y - \frac{3}{2} = -\frac{2}{3}(x - 1)$.

Similarly, the midpoint of $\overline{AC}$ is $\left(\frac{0+3}{2}, \frac{0+-2}{2}\right) = \left(\frac{3}{2}, -1\right)$ and the slope of $\overline{AC}$ is $\frac{-2-0}{3-0} = -\frac{2}{3}$. Therefore the slope of the perpendicular bisector of AC is $\frac{3}{2}$, and its equation is $y - (-1) = y + 1 = \frac{3}{2}(x - \frac{3}{2})$.

Next, convert the equations to slope intercept form:

$$y = -\frac{2}{3}x - \frac{2}{3}(-1) + \frac{3}{2} = -\frac{2}{3}x + \frac{2}{3} + \frac{3}{2} = -\frac{2}{3}x + \frac{4}{6} + \frac{9}{6} = -\frac{2}{3}x + \frac{13}{6}$$
$$y = \frac{3}{2}x - \frac{3}{2} \cdot \frac{3}{2} - 1 = \frac{3}{2}x - \frac{9}{4} - \frac{4}{4} = \frac{3}{2}x - \frac{13}{4}$$

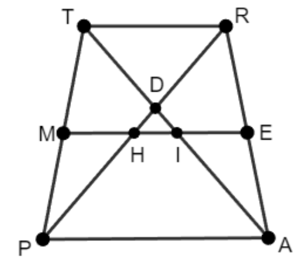
Set the two expressions equal and solve for x:

$$-\frac{2}{3}x + \frac{13}{6} = \frac{3}{2}x - \frac{13}{4}$$
$$\frac{13}{6} + \frac{13}{4} = \frac{13(2+3)}{12} = \frac{2}{3}x + \frac{3}{2}x = \left(\frac{2 \cdot 2 + 3 \cdot 3}{6}\right)x$$
$$\frac{13 \cdot 5}{12} = \frac{13}{6}x$$

Multiply both sides by $\frac{6}{13}$ and $x = \frac{5}{2}$. Plug this value for x in the second equation and $y = \frac{3}{2} \cdot \frac{5}{2} - \frac{13}{4} = \frac{15-13}{4} = \frac{1}{2}$. The desired point is therefore $\boxed{\left(\frac{5}{2}, \frac{1}{2}\right)}$.

Alternative solution: Note that the slopes of $\overline{AB}$ and $\overline{AC}$ are negative reciprocals of each other, and therefore that $\angle BAC$ is a right angle. Recall that the point that is equidistant from 3 given points is the center of the circle containing the points, so that $\angle BAC$ is an inscribed angle of the circle. Recall that twice the measure of an inscribed angle is equal to the measure of its intercepted arc. In this case, $\angle BAC$ intercepts an arc of measure $2 \cdot 90° = 180°$, that is, a semicircle. Therefore B and C lie on a diameter of the circle, and the midpoint of $\overline{BC}$ is the center of the circle, (x, y), equidistant from A, B, and C. Use the midpoint formula to find $(x, y) = (\frac{2+3}{2}, \frac{3-2}{2}) = \left(\frac{5}{2}, \frac{1}{2}\right)$.

Team Round



1. $\overline{ME}$ is the median of trapezoid $TRAP$, as shown at right. Diagonals $\overline{TA}$ and $\overline{RP}$ intersect at D within quadrilateral $TREM$. $\overline{RP}$ intersects $\overline{ME}$ at H and $\overline{TA}$ intersects $\overline{ME}$ at I. If $TR = 4$, $TD = 3$, and $DI = 1$, find PA. Express your answer as a simplified improper fraction in the form $\frac{m}{n}$.

Solution: First, note that $\overline{TR} \parallel \overline{ME} \parallel \overline{PA}$ because $\overline{ME}$ is the median of trapezoid $TRAP$ and therefore parallel to two sides. Also, note that $RE = EA$ because the median divides the opposite sides in equal parts. Therefore $TI = IA$ because three parallel lines divide any two transversals proportionately, in this case $\overline{RA}$ and $\overline{TA}$. Now $TI = TD + DI = 3 + 1 = 4$, so $IA = 4$ and $DA = DI + IA = 1 + 4 = 5$.

Note that $\angle RTA \cong \angle PAT$ and $\angle TRP \cong \angle APR$ because they are alternate interior angles created by transversals of parallel lines $\overline{RT}$ and $\overline{PA}$. Then $\triangle DTR \sim \triangle DAP$. Write the proportion: $\frac{AP}{TR} = \frac{DA}{DT}$. Substitute $TR = 4$, $DA = 5$, and $DT = TD = 3$ so that $\frac{AP}{4} = \frac{5}{3}$. Finally, solve for $PA = AP = \frac{4 \cdot 5}{3} = \boxed{\frac{20}{3}}$.

2. Mr. Smith can grade a batch of quizzes in three hours. Mr. Jones can grade a batch of quizzes in two and a half hours. If they work together at the above rates, how long will it take them to grade 3 batches of quizzes? Express your answer rounded to the nearest number of minutes.

Solution: Let s be the number of quizzes that Mr. Smith can grade in an hour and j be the number of quizzes that Mr. Jones can grade in an hour, so that $s = \frac{1}{3}$ and $j = \frac{2}{5}$. Then together Mr. S and Mr. J grade $s + j = \frac{1}{3} + \frac{2}{5} = \frac{5+3\cdot 2}{15} = \frac{11}{15}$ quizzes per hour.

Together Mr. S and Mr. J take $\frac{1}{s+j}$ hours to grade one quiz and they take $x = \frac{3}{s+j}$ hours to grade three quizzes. Therefore $x = \frac{3}{\frac{11}{15}} = 3 \cdot \frac{15}{11} = \frac{45}{11}$ hours. Convert x from hours to minutes: $60\frac{45}{11} = \frac{2700}{11} = 245\frac{5}{11}$ minutes. Round to the nearest minute: $\boxed{245}$.

3. Given the function $f(x) = \frac{1}{x}$, find the largest value of d that satisfies the equation $f(d) + f\left(\frac{1}{d}\right) = 3$. Express your answer in the simplest form $\frac{a+\sqrt{b}}{c}$.

Solution: Note that $f\left(\frac{1}{d}\right) = \frac{1}{\frac{1}{d}} = d$. Then solve, multiplying by d and rearranging to put the equation in quadratic form:

$$f(d) + f\left(\frac{1}{d}\right) = \frac{1}{d} + d = 3$$
$$1 + d^2 = 3d$$
$$d^2 - 3d + 1 = 0$$

Now apply the quadratic formula: $d = \frac{3 \pm \sqrt{(-3)^2 - 4}}{2} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$. The larger of the two solutions is $\boxed{\frac{3+\sqrt{5}}{2}}$.

4. The Groton-Dunstable A Capella club has 3 sophomores, 6 juniors, and 8 seniors. How many different five person groups can be formed from the members of the club if one sophomore, two juniors, and two seniors must be in a group?

Solution: Forming a team requires choosing 1 of 3 sophomores, 2 of 6 juniors, and 2 of 8 seniors. The number of ways to make these choices separately is $\binom{3}{1}$, $\binom{6}{2}$, and $\binom{8}{2}$ where $\binom{a}{b} = \frac{a!}{(a-b)!b!}$, the number of ways to choose b elements from a set of a distinct elements without regard to order, or "a choose b".

The choices for sophomore, juniors, and seniors are independent, so the number of five person groups is the product of the numbers of choices for the three grade levels, or $\binom{3}{1} \cdot \binom{6}{2} \cdot \binom{8}{2}$. Now

$$\binom{3}{1} = \frac{3!}{(3-1)!1!} = \frac{3 \cdot 2}{2!} = 3$$
$$\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 2} = \frac{6 \cdot 5}{2} = 15$$
$$\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{8 \cdot 7 \cdot 6!}{6!2!} = \frac{8 \cdot 7}{2} = 28$$

The total number of five person groups is the product of these three numbers: $3 \cdot 15 \cdot 28 = 3 \cdot 420 = \boxed{1260}$.

5. The three vertices of $\triangle ABC$ have coordinates $A(3, 8)$, $B(10, -6)$, and $C(-1, -8)$. Find (x, y), the coordinates of the point of intersection of the three medians of $\triangle ABC$.

Solution: The quickest way to find the coordinates is to note that the point of intersection is the center of mass of a triangle, whose coordinates are the average of the coordinates of the three vertices. Thus, $x = \frac{3+10+-1}{3} = \frac{12}{3} = 4$, and $y = \frac{8+(-6)+(-8)}{3} = \frac{-6}{3} = -2$, and $(x, y) = \boxed{(4, -2)}$.

Alternative solution: Find the equations of the lines containing two of the medians and then find the intersection of the two lines. To start, choose two vertices and find the midpoint of the opposite side for each vertex. For $B(10, -6)$, the midpoint of opposite side $\overline{AC}$ is $(\frac{3-1}{2}, \frac{8-8}{2}) = (1, 0)$. For $A(3, 8)$, the midpoint of $\overline{BC}$ is $(\frac{10-1}{2}, \frac{-6-8}{2}) = (\frac{9}{2}, -7)$.

Next, find the equations for the two medians given the vertex and the opposite side midpoint. Recall the point slope form of the equation of a line is: $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

The slope of the median containing $B(10, -6)$ and $(1, 0)$ is $m_1 = \frac{-6-0}{10-1} = \frac{-6}{9} = -\frac{2}{3}$. The equation of this first median is $y - 0 = -\frac{2}{3}(x - 1) = -\frac{2}{3}x + \frac{2}{3}$, or $y = -\frac{2}{3}x + \frac{2}{3}$.

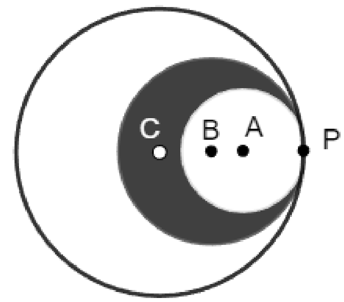
The slope of the median containing $A(3, 8)$ and $(\frac{9}{2}, -7)$ is $m_2 = \frac{8-(-7)}{3-\frac{9}{2}} = \frac{15}{-\frac{3}{2}} = -15 \cdot \frac{2}{3} = -5 \cdot 2 = -10$. The equation of this second median is $y - 8 = -10(x - 3) = -10x + 30$, $y = -10x + 30 + 8$, or $y = -10x + 38$.

Note that the two equations are in slope intercept form. Set them equal and solve for x:

$$\begin{aligned} -\frac{2}{3}x + \frac{2}{3} &= -10x + 38 \\ 10x - \frac{2}{3}x &= 38 - \frac{2}{3} \\ \left(\frac{30}{3} - \frac{2}{3}\right)x &= \frac{114}{3} - \frac{2}{3} \\ \frac{28}{3}x &= \frac{112}{3} \\ 28x &= 112, \end{aligned}$$

and $x = 4$. Use the equation for the second median to solve for y: $y = -10 \cdot 4 + 38 = -40 + 38 = -2$, confirming that $(x, y) = (4, -2)$.

6. Circles A, B, and C are internally tangent at point P, as shown at right. Let $r_A = \frac{2}{3}r_B$ and $r_B = \frac{6}{11}r_C$, where r_A, r_B, and r_C are the radii of circles A, B, and C. Find $\frac{A_S}{A_C}$, the ratio of the shaded region (A_S) to the area of circle C (A_C), expressed as a fraction.



Solution: Let A_A be the area of circle A and A_B be the area of circle B. Then area of the shaded region is the difference of these two areas, or $A_S = A_B - A_A$. The ratio to be found is $\frac{A_S}{A_C} = \frac{A_B - A_A}{A_C} = \frac{A_B}{A_C} - \frac{A_A}{A_C}$.
Recall that the ratio of the areas of two circles is the ratio of their radii squared. Thus, the ratio of the area of circle B to circle C is $\frac{A_B}{A_C} = \left(\frac{6}{11}\right)^2 = \frac{36}{121}$. Now $r_A = \frac{2}{3}r_B = \frac{2}{3} \cdot \frac{6}{11}r_C = \frac{4}{11}r_C$ and the ratio $\frac{A_A}{A_C} = \left(\frac{4}{11}\right)^2 = \frac{16}{121}$. Finally, $\frac{A_S}{A_C} = \frac{36}{121} - \frac{16}{121} = \boxed{\frac{20}{121}}$.

7. Let $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^2 + 2$. Find the value of $f^{-1}(g(f(-9)))$.

Solution: Start the evaluation from the innermost parentheses: $f(-9) = \sqrt[3]{-9+1} = \sqrt[3]{-8} = -2$. Next, find $g(f(-9)) = g(-2) = (-2)^2 + 2 = 4 + 2 = 6$.

Now find an expression for $f^{-1}(x)$. Note that $f(f^{-1}(x)) = x$. Therefore substitute $f^{-1}(x)$ for x in the expression for $f(x)$:

$$f(f^{-1}(x)) = \sqrt[3]{f^{-1}(x) + 1} = x$$

Cube both sides of this equation (take them to the third power) and:

$$\left(\sqrt[3]{f^{-1}(x) + 1}\right)^3 = f^{-1}(x) + 1 = x^3$$

Subtract 1 from both sides of the last equation to define $f^{-1}(x)$:

$$f^{-1}(x) = x^3 - 1.$$

Therefore $f^{-1}(g(f(-9))) = f^{-1}(6) = 6^3 - 1 = 216 - 1 = \boxed{215}$.

8. There are 5 green marbles and 6 red marbles in a bag. Three marbles randomly chosen from the bag. What is the probability that one marble is a different color from the other two? Write your answer as a fraction $\frac{m}{n}$.

Solution: Let P be the probability that one marble is a different color from the other two. Note that there are four possible outcomes, and that these outcomes are independent:

1. three green marbles
2. two green marbles and one red marble
3. one green marble and two red marbles
4. three red marbles

Let the probability of outcomes 1, 2, 3, and 4 be P_1, P_2, P_3, and P_4, respectively. Then $P = P_2 + P_3$. It is easier to find the probability that all three marbles are the same color than to find P. Therefore find P_1 and P_4 and then $P = 1 - (P_1 + P_4)$.

Now there are $\binom{5}{3}$, 5 choose 3, ways that three green marbles can be chosen from the five green marbles in the bag, while there are $\binom{11}{3}$ ways that any three marbles can be chosen from the eleven marbles in the bag. Therefore $P_1 = \frac{\binom{5}{3}}{\binom{11}{3}}$. Using similar reasoning, $P_4 = \frac{\binom{6}{3}}{\binom{11}{3}}$ because there are six red marbles in the bag. Calculating:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = \frac{5 \cdot 4}{2} = 10$$

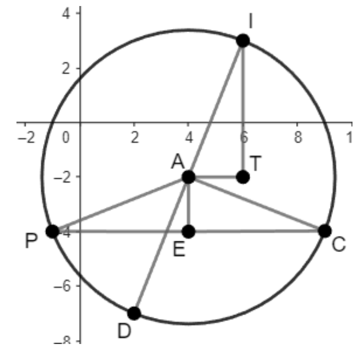
$$\binom{11}{3} = \frac{11!}{3!(11-3)!} = \frac{11 \cdot 10 \cdot 9 \cdot 8!}{3!8!} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2} = 11 \cdot 5 \cdot 3 = 55 \cdot 3 = 165$$

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 5 \cdot 4 = 20$$

Now $P_1 + P_4 = \frac{\binom{5}{3}}{\binom{11}{3}} + \frac{\binom{6}{3}}{\binom{11}{3}} = \frac{10}{165} + \frac{20}{165} = \frac{30}{165} = \frac{6}{33} = \frac{2}{11}$. Finally, $P = 1 - (P_1 + P_4) = 1 - \frac{2}{11} = \boxed{\frac{9}{11}}$.

9. Circle A has a diameter with endpoints $(2, -7)$ and $(6, 3)$. The point $(x, -4)$ is on the circle. Find all possible values of x.

Solution: First draw the circle on the coordinate axes, as shown below, right.



The endpoints of the diameter are labeled $D(2, -7)$ and $I(6, 3)$. The center of circle A is the midpoint of $\overline{DI}$. Using the midpoint formula, the coordinates of A are $\left(\frac{2+6}{2}, \frac{-7+3}{2}\right) = (4, -2)$.
Note that any point on the circle with coordinates $(x, -4)$ lies on the line $y = -4$. Note also that there are two possible values for x corresponding to the two points of intersection of the line with the circle, labeled P and C in the figure. The midpoint of $\overline{PC}$ is labeled E, with coordinates $(4, -4)$: $y = -4$ because E lies on $\overline{PC}$ and $x = 4$ because $\overline{AE} \perp \overline{PC}$.

Next, place point $T(6, -2)$ to construct triangle $\triangle AIT$, which is a right triangle: $\overline{TA}$ lies on the line $y = -2$ and is parallel to the x-axis and $\overline{IT}$ lies on the line $x = 6$ and is parallel to the y-axis. Note that $AE = 2 = AT$, so that right triangles AIT, ACE, and APE are congruent to each other by the Hypotenuse Leg Theorem. Therefore $CE = PE = IT = 3 - (-2) = 3 + 2 = 5$ (difference of the y-coordinates of I and T). Thus the possible values for x are $4 \pm 5 = \boxed{-1, 9}$.